Last time: $S_\lambda \cap \overline{Gr}_\mu \neq \emptyset \iff t^\lambda \in \overline{Gr}_\mu$

1. Iwahori
2. SL_2
3. $Gr_{GL_n, \leq (n,0,\dots,0)}$

$I = ev^{-1}(B)$, $ev : L^{\geq 0}G \to G$.

G_F Kac-Moody group $\mathfrak{g}_F$

T $\quad \mathfrak{g}_{\alpha,i}$, $\alpha \in \Delta(G,T)$, $i \in \mathbb{Z}$. $\qquad \alpha \in \Delta(G,T) \sqcup \{0\}$

positive roots $=$ $\Delta^+(G,T) \sqcup \{(\alpha,i) : i > 0\}$.

Parahori group $\supseteq I$.

eg. $L^{\geq 0}G = L^{>0}G \rtimes G$ $\quad$ Levi decomposition

$N = N_{G_F}(T) = N_G(T)_F$.

$\widetilde{W} = N/N\cap B = N_G(T)_F/T_0 = X_\cdot(T) \rtimes W$.

$I \backslash G_F / I = \widetilde{W}$

$L^{\geq 0}G \backslash G_F / L^{\geq 0}G = W\backslash \widetilde{W}/W \xleftrightarrow{1:1} X_\cdot^+$

$^gH = gHg^{-1}$

$I^{<\lambda} = I \cap {}^{t^\lambda}L^{<0}G \qquad I^{\geq\lambda} = I \cap {}^{t^\lambda}L^{\geq 0}G$.

$L^{<0}G \times L^{\geq 0}G \to LG$ open

$I^{<\lambda} \times I^{\geq\lambda} \to I$ open

$I \times U^- \to L^{\geq 0}G$ open

$$\begin{tikzcd}
I^{<\lambda} \times \underbrace{I^{\geq\lambda} \times U^-} \arrow[r] \arrow[d] & L^{\geq 0}G \text{ open} \arrow[d, "L^{\geq 0} \cap {}^{t^\lambda}L^{\geq 0}"] \\
I^{<\lambda} \arrow[r, hook] & Gr_\lambda \text{ open}
\end{tikzcd}$$

$P, Q \quad Q = U \rtimes L, \qquad P \cap {}^{w}Q = (P \cap {}^{w}U) \rtimes (P \cap {}^{w}L)$

$G \cap {}^{t^\lambda} L^{\geq 0} = P_\lambda$

$g \in G, \quad g = u w p, \quad u \in U_\lambda^+, w \in W_\lambda, p \in P_\lambda.$

$t^{-\lambda} u w t^{\lambda} \in L^{\geq 0}G$

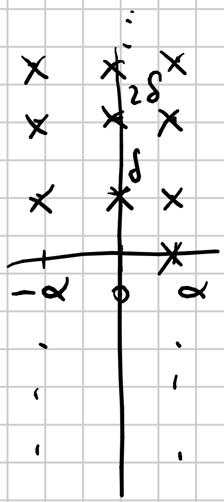
$$(t^{-\lambda} u t^{\lambda}) \cdot \underbrace{t^{-\lambda} w t^{\lambda} w^{-1}}_{\substack{\| \\ t^{w\lambda - \lambda}}} \cdot w \qquad w \in G \subseteq L^{\geq 0}G$$

$$(t^{-\lambda} u t^{\lambda}) \cdot t^{w\lambda-\lambda} \in L^{\geq 0}G$$

$\Rightarrow t^{-\lambda} u t^{\lambda}, \ t^{w\lambda - \lambda} \in L^{\geq 0}G$

$\Rightarrow w = 1, \ u = 1 \qquad \square$

2. $SL_2 \qquad X_\cdot(T) = \mathbb{Z}\alpha^\vee \qquad$ weights $\mathbb{Z}\alpha \oplus \mathbb{Z}\delta$



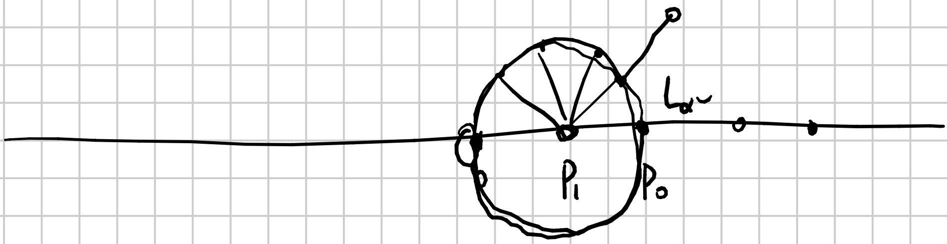
simple: $\alpha, \delta - \alpha$

$\alpha \to P_0 = SL_{2,\mathcal{O}}$

$\delta - \alpha \to P_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_{2,F} : a, d \in \mathcal{O}, \ c \in t\mathcal{O}, \ b \in t^{-1}\mathcal{O} \right\}$

$I \subseteq P_0, \ I \leq P_1$

$P_1/I, \quad P_1/I \simeq \mathbb{P}^1$



$t^{\lambda - \alpha^\vee} \in \overline{Gr}_\lambda$

$\lambda = \alpha^\vee$, $t^\lambda = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \in G_F$.

$G_{\mathcal{O}} t^\lambda = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_{2,F} : \begin{array}{l} a, c \in t\mathcal{O} \\ b, d \in t^{-1}\mathcal{O} \end{array} \right\}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathcal{O}} \cap t^\lambda G_{\mathcal{O}} t^{-\lambda}$, i.e. $t^{-\lambda} \begin{pmatrix} a & b \\ c & d \end{pmatrix} t^\lambda \in G_{\mathcal{O}}$

$$t^{-\lambda} \begin{pmatrix} a & b \\ c & d \end{pmatrix} t^\lambda = \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = \begin{pmatrix} t^{-1}a & t^{-1}b \\ tc & td \end{pmatrix} \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = \begin{pmatrix} a & t^{-2}b \\ t^2c & d \end{pmatrix}$$

$\Rightarrow \quad b \in t^2\mathcal{O}$

$G_{\mathcal{O}} / \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : b \in t^2\mathcal{O} \right\}$

3. $G = GL_n$, $\omega_n = (n, 0, \ldots, 0)$

$X = \mathbb{Z}^n$. positive $e_i - e_j$, $i > j$.

$Gr_{\leq \omega_n} = \{ \Lambda \subset \Lambda_0 : \dim \Lambda_0/\Lambda = n \}$, $\Lambda_0 = k[[t]]^n$.

$\lambda \leq \omega_n$, $\lambda = (\lambda_1, \ldots, \lambda_n) : \sum \lambda_i = n$.

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.

$t^\lambda \to t^{\lambda_1}\mathcal{O} \oplus t^{\lambda_2}\mathcal{O} \oplus \cdots \oplus t^{\lambda_n}\mathcal{O}$

$Gr_{\leq \omega_n} \subseteq \text{RHS}$, $t^\lambda \in \text{RHS} \Rightarrow \lambda \leq \omega_n$.

N_n = nilpotent $n \times n$ matrix,

$N_n \to Gr_{\leq \omega_n}$ locally closed

$A \mapsto (t - A)\Lambda_0$

$\dim N_n = n(n-1) = \langle 2\rho, \omega_n \rangle$

$L^{<0}G \overset{\text{open}}{\hookrightarrow} Gr$

$\Rightarrow$ open

$1 - t^{-1}A$

$$\mathrm{Gr}_{\le\lambda} \tilde{\times} \mathrm{Gr}_{\le\mu} \to \mathrm{Gr}_{\le\lambda+\mu} \qquad \omega_1 = (1, 0, \dots, 0)$$

$$\mathrm{Gr}_{\le\omega_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_{\le\omega_1} \to \mathrm{Gr}_{\le\omega_n}$$

① $$\mathrm{Gr} \tilde{\times} \mathrm{Gr} = G_F \times^{G_O} \mathrm{Gr} \xrightarrow{m} \mathrm{Gr}$$

② $$(\mathrm{Gr} \tilde{\times} \mathrm{Gr})(R) = \left\{ (\mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) : \begin{array}{l} \mathcal{E}_1, \mathcal{E}_2 : G\text{-torsor} \\ \beta_1 : \mathcal{E}_1|_{D_R^*} \to \mathcal{E}_2|_{D_R^*} \\ \beta_2 : \mathcal{E}_2|_{D_R^*} \to \mathcal{E}_0|_{D_R^*} \end{array} \right\}$$

$$\downarrow \qquad\qquad \mapsto$$

$$\mathrm{Gr} \qquad\qquad (\mathcal{E}_1, \beta_2\beta_1)$$

$$\begin{array}{ccc} \tilde{N}_n & \longrightarrow & \mathrm{Gr}_{\le\omega_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_{\le\omega_1} \\ \downarrow & \square & \downarrow \\ N_n & \longrightarrow & \mathrm{Gr}_{\le\omega_n} \end{array}$$

$\mathrm{Gr}_{\le\omega_1} = \{\Lambda \subset \Lambda_0 ; \dim \Lambda_0/\Lambda = 1\}$

$\mathrm{Gr}_{\le\omega_n} \ni \Lambda_n , \dim \Lambda_0/\Lambda_n = n$

$$\text{fiber} = \left\{ \Lambda_n \subset \Lambda_{n-1} \subset \cdots \subset \Lambda_1 \subset \Lambda_0 : \dim \Lambda_i/\Lambda_{i+1} = 1 \right\}$$

$X = \mathbb{A}^1$,

$$\begin{array}{ccccc} \tilde{\mathfrak{gl}}_n & \to & \mathrm{Gr}_{\le\omega_1, X} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_{\le\omega_1, X} & \to & X^n \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{gl}_n & \to & \mathrm{Gr}^{(n)}_{\le\omega_1} & \longrightarrow & X^n/\mathfrak{S}_n \end{array}$$

generalize to